6. THE FINITE SQUARE WELL

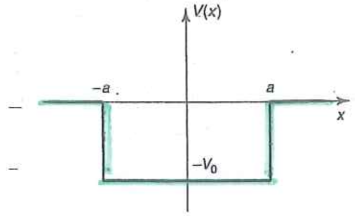

FIGURE 2.17: The finite square well (Equation 2.145).

$$V(x) = \begin{cases} -V_0 & -a \le x \le a \\ 0 & |x| > a \end{cases} \qquad (V_0 > 0)$$

BOUND STATES $(-V_0 < E < 0)$: $x < -a$ $\quad V(x) = 0 \; ; \; \dfrac{d^2\psi}{dx^2} = \kappa^2 \psi \qquad \kappa = \dfrac{\sqrt{-2mE}}{\hbar} > 0$

$$\psi(x) = \cancel{A}\, e^{-\kappa x} + B e^{+\kappa x} = B e^{\kappa x}$$

($A \to 0$, DIVERGES)

BUT $E + V_0 > 0$ ENERGY DIFF FROM BOTTOM OF WELL

$-a < x < a$ $\quad V(x) = -V_0 \; ; \; -\dfrac{\hbar^2}{2m}\dfrac{d^2\psi}{dx^2} - V_0\psi = E\psi \; ; \; \dfrac{d^2\psi}{dx^2} = -\ell^2\psi \; ; \; \ell = \dfrac{\sqrt{2m(E+V_0)}}{\hbar}$

$$\psi(x) = C \sin(\ell x) + D \cos(\ell x)$$

(BETTER THAN EXPONENTIALS $e^{\pm i\ell x}$ DUE TO SYMMETRY)

$x > a$ SIMILARLY

$$\psi(x) = F e^{-\kappa x} + \cancel{G}\, e^{+\kappa x} = F e^{-\kappa x}$$

($G \to 0$, DIVERGES)

BOUNDARY CONDITIONS

~~DUE TO SYMMETRIC POTENTIAL ⇒ SOLUTIONS ARE EVEN OR ODD~~ $\begin{cases} \text{EVEN} & \psi(-x) = \psi(x) \\ \text{ODD} & \psi(-x) = -\psi(x) \end{cases}$

LET'S DO ONLY EVEN SOLUTIONS (ODD SOLUTIONS IN PROBLEM 2.29)

JUST ENFORCE THEM ON ONE SIDE

$$\psi(x) = \begin{cases} F e^{-\kappa x} & x > a \\ D \cos(\ell x) & 0 < x < a \\ \psi(-x) & x < 0 \quad \text{(FOR SYMMETRY REASONS)} \end{cases}$$

CONTINUITY AT $x = a$ OF $\psi(x)$

$$F e^{-\kappa a} = D \cos(\ell a)$$

OF $\dfrac{d\psi(x)}{dx}$

$$-\kappa F e^{-\kappa a} = -\ell D \sin(\ell a)$$

DIVIDE THROUGH

$$\boxed{\kappa = \ell \tan(\ell a)}$$

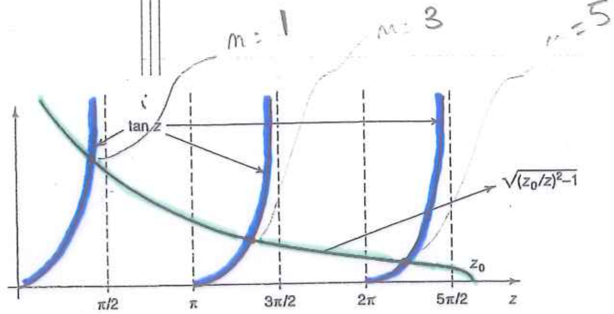

FIGURE 2.18: Graphical solution to Equation 2.156, for $z_0 = 8$ (*even* states).

EVEN ONLY

$$\kappa = \ell \tan(\ell a)$$

$z \equiv \ell a$ $\qquad \kappa = \frac{\sqrt{-2mE}}{\hbar}$

$z_0 \equiv \frac{a}{\hbar}\sqrt{2mV_0}$ $\qquad \ell = \frac{\sqrt{2m(E+V_0)}}{\hbar}$

$$\kappa^2 + \ell^2 = -\frac{2mE}{\hbar^2} + \frac{2mE}{\hbar^2} + \frac{2mV_0}{\hbar^2} = \frac{2mV_0}{\hbar^2}$$

$$\Rightarrow \tan(\ell a) = \tan z = \frac{\kappa}{\ell} = \frac{\kappa a}{\ell a} = \frac{\sqrt{z_0^2 - z^2}}{z} = \sqrt{\left(\frac{z_0}{z}\right)^2 - 1}$$

~~START HERE~~

$$\left(\text{SINCE } \kappa a = \sqrt{z_0^2 - z^2} = \sqrt{\frac{2mV_0}{\hbar^2}a^2 - \ell^2 a^2} = \sqrt{\kappa^2 a^2 + \ell^2 a^2 - \ell^2 a^2} = \kappa a\right)$$

$$\Rightarrow \boxed{\tan z = \sqrt{(z_0/z)^2 - 1}}$$

$z_0 \equiv \frac{a}{\hbar}\sqrt{2mV_0}$ MEASURES SIZE OF WELL

$z = \ell a = \frac{\sqrt{2m(E+V_0)}}{\hbar}a$ RELATED TO E

$E < 0$

CAN BE SOLVED NUMERICALLY OR GRAPHICALLY AS IN FIGURE

ONLY A FINITE # OF BOUND STATES

$$E + V_0 = \frac{\hbar^2 z^2}{2ma^2}$$

SPECIAL CASES:

1) WIDE, DEEP WELL

z_0 VERY LARGE → INTERSECTION SLIGHTLY BELOW $z_m \cong m\pi/2$ $\quad$ m ODD

$m = 1, 3, 5$

$$\Rightarrow E_m + V_0 \cong \frac{m^2\pi^2\hbar^2}{2m(2a)^2}$$

ENERGY ABOVE BOTTOM OF WELL

INFINITE SQUARE WELL ENERGIES FOR WIDTH 2a (HALF SOLUTIONS FOR EVEN WAVE FCTS)

2) SHALLOW, NARROW WELL

FOR z_0 DECREASING → FEWER BOUND STATES

UNTIL FOR $z_0 < \pi/2$ ONLY ONE BOUND STATE REMAINS → $m = 1$

(2)

(USE PHET DEMOS...)

SCATTERING STATES **$E > 0$**

$x < -a$: $\psi(x) = A e^{ikx} + B e^{-ikx}$ $\qquad k = \dfrac{\sqrt{2mE}}{\hbar}$

$V(x) = 0$

$-a < x < a$: $\psi(x) = C \sin(\ell x) + D \cos(\ell x)$ $\qquad \ell = \dfrac{\sqrt{2m(E+V_0)}}{\hbar}$

$V(x) = -V_0$

$x > a$: $\psi(x) = F e^{ikx} + \left(G e^{-ikx}\right)$

OMITTED

$V(x) = 0$

CONTINUITY OF ψ AT $-a$: $A e^{-ika} + B e^{ika} = -C\sin(\ell a) + D\cos(\ell a)$

CONTINUITY OF $\frac{d\psi}{dx}$ AT $-a$: $ik\left[A e^{-ika} - B e^{ika}\right] = \ell\left[C\cos(\ell a) + D\sin(\ell a)\right]$

CONTINUITY OF ψ AT $+a$: $C\sin(\ell a) + D\cos(\ell a) = F e^{ika}$

CONTINUITY OF $\frac{d\psi}{dx}$ AT $+a$: $\ell\left[C\cos(\ell a) - D\sin(\ell a)\right] = ik F e^{ika}$

USE TWO EQTS TO ELIMINATE C, D SOLVE FOR B AND F (PR. 2.32)

...

$$B = i\,\frac{\sin(2\ell a)}{2k\ell}\left(\ell^2 - k^2\right) F$$

SKIP ALGEBRA

TRANSMISSION COEFFICIENT

$$F = \frac{e^{-2ika} A}{\cos(2\ell a) - i\frac{(k^2+\ell^2)}{2k\ell}\sin(2\ell a)} \Rightarrow T = \frac{|F|^2}{|A|^2}$$

$$\Rightarrow T^{-1} = 1 + \frac{V_0^2}{4E(E+V_0)}\sin^2\left(\frac{2a}{\hbar}\sqrt{2m(E+V_0)}\right)$$

$T^{-1} = 1$ IF $\sin(\;) = 0 \Rightarrow \dfrac{2a}{\hbar}\sqrt{2m(E+V_0)} = m\pi$

TRANSPARENT WELL

EXACT FORMULA

$$\Rightarrow E_m + V_0 = \frac{m^2\pi^2\hbar^2}{2m(2a)^2}$$

SAME AS ALLOWED ENERGIES FOR INFINITE SQUARE WELL (BUT POSITIVE E_m IN THIS CONTEXT)

ENERGIES FOR PERFECT TRANSMISSION

(RAMSAUER-TOWNSEND EFFECT)

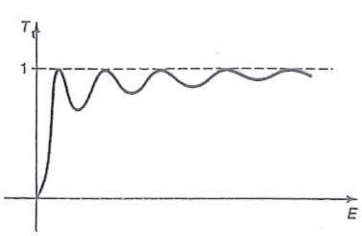

FIGURE 2.19: Transmission coefficient as a function of energy (Equation 2.169).