Applications of Fractional Calculus to Newtonian Mechanics

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Introduction

Fractional Calculus (FC) is a natural generalization of calculus that studies the possibility of computing derivatives and integrals of any real (or complex) order, i.e., not just of standard integer orders, such as first-derivative, second-derivative, etc.

The history of FC started in 1695 when l'Hôpital raised the question as to the meaning of taking a fractional derivative such as $d^{1/2}y/dx^{1/2}$ and Leibniz replied: "...This is an apparent paradox from which, one day, useful consequences will be drawn."

Since then, eminent mathematicians such as Fourier, Abel, Liouville, Riemann, Weyl, Riesz, and many others contributed to the field, but until lately FC has played a negligible role in physics.

However, in recent years, applications of FC to physics have become more common in fields ranging from classical and quantum mechanics, nuclear physics, hadron spectroscopy, and up to quantum field theory.

A brief review of FC

Unlike standard calculus, there is no unique definition of derivation and integration in FC. Historically, several different definitions were introduced and used.

All proposed definitions reduce to standard derivatives and integrals for integer orders n, but they might not be fully equivalent for non-integer orders of differ-integration. For example, Riemann proposed to extend the simple recursive relation $\frac{d^n x^k}{dx^n} = \frac{k!}{(k-n)!}x^{k-n}$ by using gamma functions: $\frac{d^q x^k}{dx^q} = \frac{\Gamma(k+1)}{\Gamma(k-q+1)}x^{k-q}$, where q is a real (or complex) number. In this way, for any function f(x) expanded in series of power functions, $f(x) = \sum_{k=0}^{\infty} a_k x^k$, one can easily define a "fractional derivative" of arbitrary order q: $\frac{d^q f(x)}{dx^q} = \sum_{k=0}^{\infty} a_k \frac{\Gamma(k+1)}{\Gamma(k-q+1)}x^{k-q}$. We note that, using this definition of fractional derivative, the derivative of a constant C is not zero in FC: $\frac{d^q C}{dx^q} = \frac{d^q (Cx^0)}{dx^q} = \frac{Cx^{-q}}{\Gamma(1-q)}$.

More general definitions of fractional derivatives and integrals to any arbitrary order q exist in the literature, such as the Grünwald formula:

$$\frac{d^q f}{[d(x-a)]^q} = \lim_{N\to\infty}\left\{\frac{\left[\frac{x-a}{N}\right]^{-q}}{\Gamma(-q)}\sum_{j=0}^{N-1}\frac{\Gamma(j-q)}{\Gamma(j+1)}f\left(x-j\left[\frac{x-a}{N}\right]\right)\right\},$$

which involves only evaluations of the function itself and can be used for both positive (derivation) and negative values of q (integration). Another general definition is the Riemann-Liouville fractional integral, for negative q: $\frac{d^q f}{[d(x-a)]^q} = \frac{1}{\Gamma(-q)}\int_a^x (x-y)^{-q-1} f(y)\,dy \quad (q<0)$, which can also be extended to fractional derivatives.

Generalizing Mechanics

One-dimensional Newtonian mechanics for a point-particle of constant mass m is based upon Newton's second law of motion, $\frac{d^2x(t)}{dt^2} = \frac{F}{m}$. As our first example of FC applied to mechanics, we generalize the previous equation by using derivatives of arbitrary (real) order q, and by considering a constant force per unit mass, f=F/m=const: $\frac{d^q x(t)}{dt^q} = \frac{F}{m} = f$.

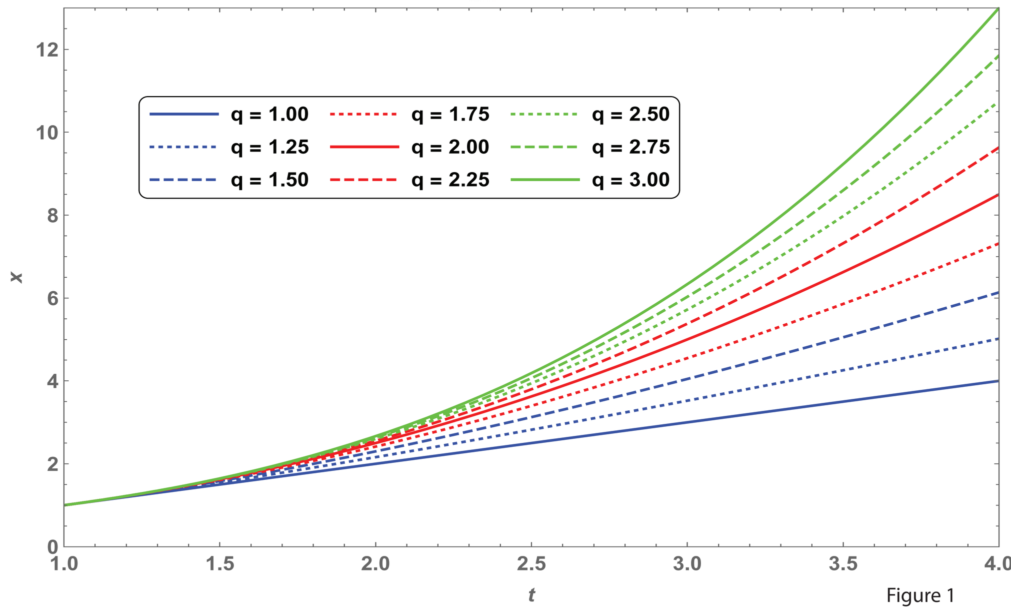


Figure 1

The general solution of the (extraordinary) differential equation above is:

$$x(t) = \frac{d^{-q}f}{dt^{-q}} + c_1 t^{q-1} + c_2 t^{q-2} + \ldots + c_l t^{q-l}$$
$$= \frac{f t^q}{\Gamma(1+q)} + c_1 t^{q-1} + c_2 t^{q-2} + \ldots + c_l t^{q-l}$$
$$\text{with} \left\{ \begin{array}{c} 0 < q \le l < q+1, \text{if } q > 0 \\ l = 0, \text{ if } q \le 0 \end{array} \right\},$$

where we also used the fractional derivative of a constant quantity f, which is not zero in FC.

The l constants of integration, c_1, c_2, ... , c_l, can be determined from the l initial conditions: $x(t_0)$, $x'(t_0)$,..., $x^{(l-1)}(t_0)$. For example, choosing for simplicity's sake $t_0=1$, $x(1)=x'(1)=x''(1)=...=1$, and also f=1, our general solution above can be easily plotted for different values of the order q, as shown in Figure 1 in the panel above.

This figure illustrates the resulting position vs. time functions, with the order q ranging from 1 to 3 with fractional increments. The standard Newtonian solution is obviously recovered for q=2 (red-solid curve), for a motion with constant acceleration. Two other solutions for integer values of q are presented: the case for q=1 (blue-solid line) represents a simple motion with constant velocity; the q=3 case (green-solid line) represents instead a motion with constant jerk. In Fig.1, we also show (dashed and dotted curves) the position vs. time functions for some fractional values of the order q. These additional curves interpolate well between the integer-order functions described above.

Gravitational Force and FC

Generalized Riesz Gravitational Potential

A possible generalization of Newton's law of universal gravitation can be obtained by considering a generalized Riesz potential: $V_{RZ}(\vec{r}) = -\frac{G}{a}\int_{\mathbb{R}^3}\frac{dM}{(s/a)^q} = -\frac{G}{a}\int_{\mathbb{R}^3}\frac{\rho(\vec{r}')d^3\vec{r}'}{\left(\left|\vec{r}-\vec{r}'\right|/a\right)^q}$, where $s = \left|\vec{r}-\vec{r}'\right|$ is the distance between the infinitesimal source mass element $dM = \rho(\vec{r}')d^3\vec{r}'$ and the position $\vec{r}$ being considered.

Due to the fractional order q, a "length scale" a is needed for dimensional correctness. For a spherical source of radius R_0 and uniform density $\rho_0 = M/(\frac{4}{3}\pi R_0^3)$, the Riesz potential can be evaluated analytically for any q:

$$V_{RZ}(r) = -\frac{G\rho_0}{r}\frac{2\pi a^{q-1}}{(q-2)(q-3)(q-4)}$$
$$\times \left\{ \begin{array}{l} \left[\begin{array}{c} (r+R_0)^{3-q}(r-(3-q)R_0) \\ +(R_0-r)^{3-q}(r+(3-q)R_0) \end{array} \right], \text{ for } 0 \le r \le R_0 \\ \left[\begin{array}{c} (r+R_0)^{3-q}(r-(3-q)R_0) \\ -(r-R_0)^{3-q}(r+(3-q)R_0) \end{array} \right], \text{ for } r > R_0 \end{array} \right\}.$$

In Figure 2, we illustrate the shape of these generalized gravitational potentials, for different values of the fractional order q ranging from zero to two. The q=1 case (red-solid curve) represents the standard Newtonian gravitational potential. All these plots were obtained by setting G = M = R_0 = a = 1 for simplicity's sake, thus the vertical grid line at r=1.0 denotes the boundary between the inner ($0 \le r \le R_0$) and the outer ($r > R_0$) potentials.

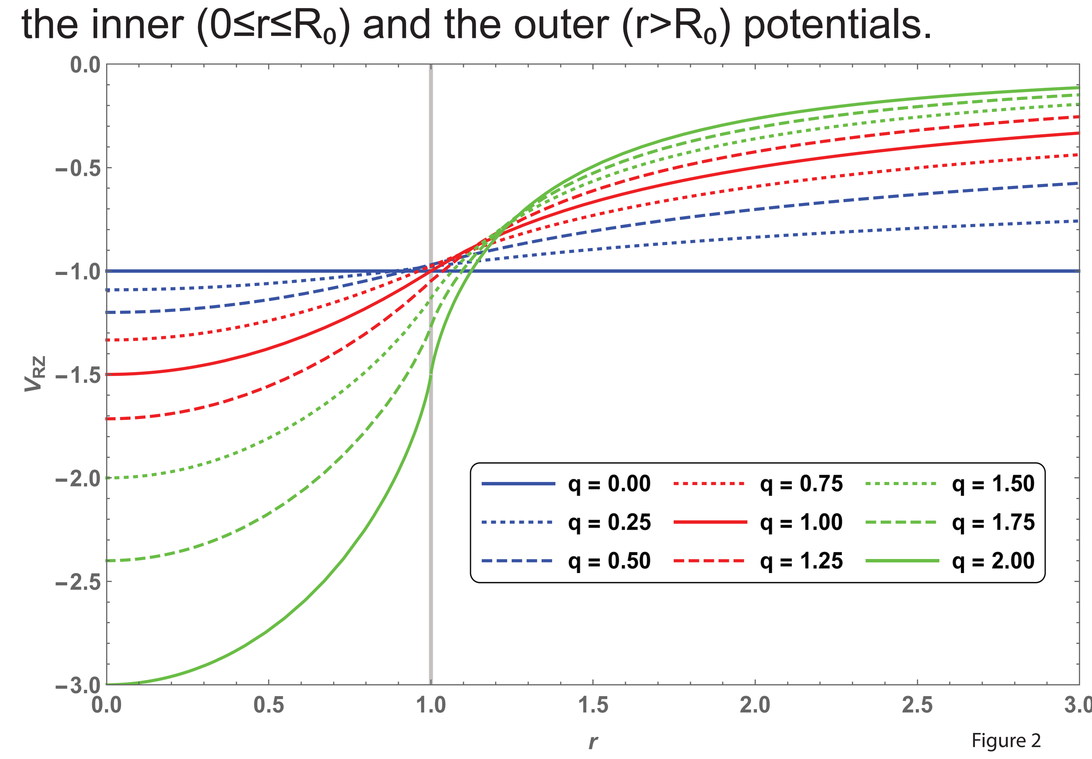


Figure 2

Orbital Circular Velocities

An interesting consequence of these generalized gravitational potentials is the analysis of the resulting orbital circular velocities, for the inner and outer solutions. From the Riesz potentials, we can easily obtain the circular velocities,

$$v_{circ}(r) = \sqrt{r\frac{\left|\vec{F}_{RZ}(r)\right|}{m}} = \sqrt{\left|r\frac{dV_{RZ}(r)}{dr}\right|},$$

which are plotted in Figure 3, for the same values of the fractional order q used in Fig. 2. We also set G=M=R_0=a=1 as done previously. The q=1 case (red-solid curve) represents the standard Newtonian situation.

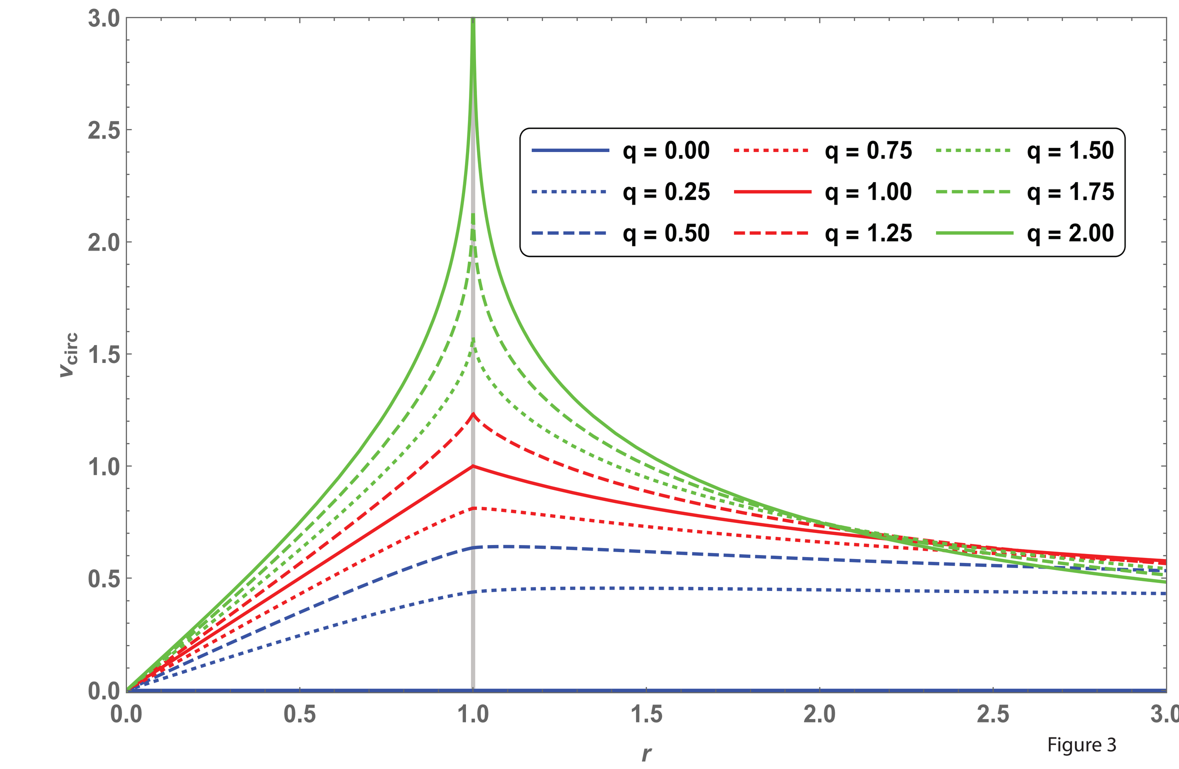


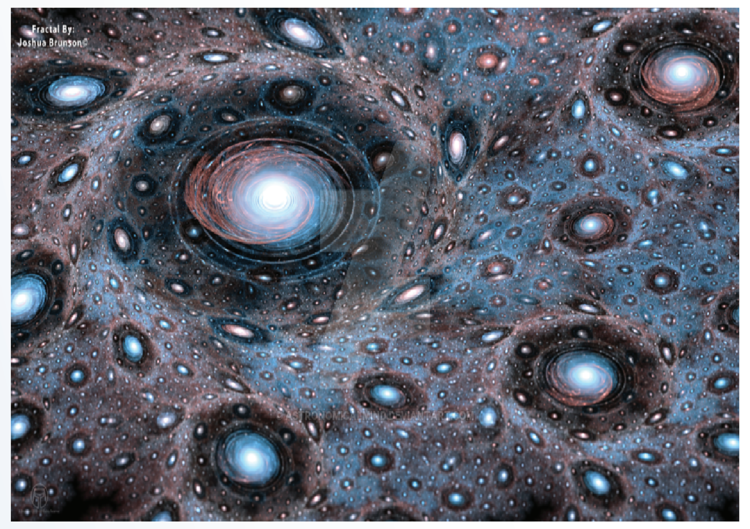
Figure 3

It is interesting to note that, for values of q decreasing from one toward zero, the rotational velocity curves in the outer ($r \ge R_0$) region show a definite "flattening" effect, which becomes more pronounced for the lowest q values (for example, in the q=0.25 case, blue-dotted curve). This consideration might be of some interest in relation with the well-established problem of dark matter in galaxies, as evidenced by the galactic rotation curves and their lack of Newtonian behavior in the outer regions.

Also, given possible connections between fractional calculus and fractal geometry, the self-similar patterns shown by mathematical fractals might also be present in the astrophysical structures of the Universe, such as galaxies or others. The fractal dimension of these structures can be directly connected to the fractional order of the related gravitational equations.

Conclusions

In this work, we have applied Fractional Calculus to some basic problems in standard Newtonian mechanics. The main goal was to show that FC can be used as a pedagogical tool, even in introductory physics courses, to gain more insight into basic concepts of physics, such as Newton's laws of motion and gravitation.



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